

HOPF DECOMPOSITION AND HOROSPHERIC LIMIT SETS

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ABSTRACT. By looking at the relationship between the recurrence properties of a countable group action with a quasi-invariant measure and the structure of its ergodic components we establish a simple general description of the Hopf decomposition of the action into the conservative and the dissipative parts in terms of the Radon–Nikodym derivatives of the action. As an application we prove that the conservative part of the boundary action of a discrete group of isometries of a Gromov hyperbolic space with respect to any invariant quasi-conformal stream coincides (mod 0) with the big horospheric limit set of the group.

Conservativity and dissipativity are, alongside with ergodicity, the most basic notions of the ergodic theory and go back to its mechanical and thermodynamical origins. The famous *Poincaré recurrence theorem* states that any invertible transformation T preserving a probability measure m on a state space X is *conservative* in the sense that any positive measure subset $A \subset X$ is *recurrent*, i.e., for a.e. starting point $x \in A$ the trajectory $\{T^n x\}$ eventually returns to A . These definitions obviously extend to an arbitrary measure class preserving action $G \curvearrowright (X, m)$ of a general countable group G on a probability space (X, m) . The opposite notions are those of *dissipativity* and of a *wandering set*, i.e., such a set A that all its translates gA , $g \in G$, are pairwise disjoint. An action is called *dissipative* if it admits a positive measure wandering set, and it is called *completely dissipative* if, moreover, there is a wandering set such that the union of its translates is (mod 0) the whole action space.

Our approach to these properties is based on the observation that the notions of conservativity and dissipativity admit a very natural interpretation in terms of the *ergodic decomposition* of the action (under the assumption that such a decomposition exists, i.e., the action space is a *Lebesgue measure space*). Let $\mathcal{C} \subset X$ denote the union of all the purely non-atomic components, and let $\mathcal{D} = X \setminus \mathcal{C}$ be the union of all the purely atomic ergodic components. We call \mathcal{C} and \mathcal{D} the *continual* and *discontinual* parts of the action, respectively. Further, let $\mathcal{D}_{\text{free}}$ be the subset of \mathcal{D} consisting of the points with trivial stabilizers, i.e., the union of free orbits in \mathcal{D} . The restriction of the action to the set $\mathcal{C} \cup (\mathcal{D} \setminus \mathcal{D}_{\text{free}})$ is conservative, whereas the restriction to the set $\mathcal{D}_{\text{free}}$ is completely dissipative, thus providing the so-called *Hopf decomposition* of the action space into the conservative and completely dissipative parts (Theorem 11). [Historically, such a decomposition was first established in the pioneering paper of Eberhard Hopf [Hop30] for one-parameter groups of measure preserving transformations.] Although this fact is definitely known to the specialists, it rather belongs to the “folklore”, and the treatment of this issue in the literature is sometimes pretty confused, so that we felt it necessary to give a clear and concise proof.

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The continual part \mathcal{C} can be described as the set of points for which the orbitwise sum of the Radon–Nikodym derivatives of the action is infinite (Theorem 19(iii)). Therefore, in the case of (mod 0) free actions this condition completely characterizes the conservative part of the action. Once again, the specialists in the theory of discrete equivalence relations will hardly be surprised by this result. However, to the best of our knowledge, in spite of its simplicity it has never been formulated explicitly. A simple consequence of this fact is the description of the conservative part of a free action $G \curvearrowright (X, m)$ as the set of points $x \in X$ with the property that

(*) there exists $t = t(x)$ such that $\{g \in G : dgm/dm(x) \geq t\}$ is infinite

(Theorem 19(iv)).

The latter result completely trivializes the problem of a geometric description of the conservative part of the boundary action of a discrete group of isometries G of a Gromov hyperbolic space \mathcal{X} with respect to a certain natural measure class, which is our main application.

More precisely, for any boundary point $\omega \in \partial\mathcal{X}$ and any $x, y \in X$ let $\beta_\omega(x, y) = \limsup_z [d(y, z) - d(x, z)]$, where $z \in \mathcal{X}$ converges to ω in the hyperbolic compactification. For $\text{CAT}(-1)$ spaces β_ω are the usual *Busemann cocycles*, whereas in the general case the cocycle identity is satisfied up to a uniformly bounded additive error only, so that we have to call them *Busemann quasi-cocycles*. Then one can look for a family $\lambda = \{\lambda_x\}$ of pairwise equivalent finite boundary measures parameterized by points $x \in \mathcal{X}$ (following [KL05] we use the term *stream* for such families) whose mutual Radon–Nikodym derivatives are prescribed by β_ω in the sense that

$$(**) \quad \left| \log \frac{d\lambda_x}{d\lambda_y}(\omega) - D\beta_\omega(x, y) \right| \leq C \quad \forall x, y \in \mathcal{X}, \omega \in \partial\mathcal{X}$$

for certain constants $C \geq 0, D > 0$. We call such a stream *quasi-conformal of dimension D* . It is *invariant* if $g\lambda_x = \lambda_{gx}$ for any $g \in G, x \in \mathcal{X}$.

In the $\text{CAT}(-1)$ case any invariant quasi-conformal stream is equivalent (with uniformly bounded Radon–Nikodym derivatives) to an invariant conformal *stream* of the same dimension, i.e., such that the logarithms of the Radon–Nikodym derivatives are precisely proportional to the Busemann cocycles. Given a reference point $o \in \mathcal{X}$, an invariant (quasi-)conformal stream is uniquely determined just by a (*quasi*-)conformal measure λ_o with the property that $\log dg\lambda_o/d\lambda_o(\omega)$ is proportional to $\beta_\omega(go, o)$ (up to a uniformly bounded additive error).

Coornaert [Coo93] proved (applying the construction used by Patterson [Pat76] in the case of Fuchsian groups) that for any discrete group of isometries G of a Gromov hyperbolic space \mathcal{X} with a finite critical exponent D there exists an invariant quasi-conformal stream of dimension D supported by the limit set of G .

It is with respect to the measure class of an invariant quasi-conformal stream $\lambda = \{\lambda_x\}$ that we study the Hopf decomposition of the boundary action. The geometric description (**) of the Radon–Nikodym derivatives in combination with criterion (*) immediately implies that the conservative part of the action coincides (mod 0) with the *big horospheric limit set* Λ_G^{horB} of the group G , i.e., the set of points $\omega \in \partial\mathcal{X}$ for which there exists $t = t(\omega)$ such that $\{g \in G : \beta_\omega(go, o) \geq t\}$ is infinite for a certain fixed reference point $o \in X$, or, in other words, the set of points $\omega \in \partial\mathcal{X}$ such that a *certain* horoball centered at ω contains infinitely many points from the orbit Go (Theorem 33).

This characterization of the conservative part of the boundary action was first established by Pommerenke [Pom76] for Fuchsian groups with respect to the visual stream on

the boundary circle (although in a somewhat different terminology). Pommerenke's argument uses analytic properties of the Blaschke products and does not immediately carry over to the higher dimensional situation. Sullivan [Sul81] used a more direct geometrical approach and proved this characterization for Kleinian groups, again with respect to the visual stream. Actually he considered the *small horospheric limit set* Λ_G^{horS} (also called just *horospheric limit set*; it is defined by requiring that the intersection of *any* horoball centered at $\omega \in \Lambda_G^{\text{horS}}$ with the orbit Go be infinite) essentially showing that $\Lambda^{\text{horB}} \setminus \Lambda^{\text{horS}}$ is a null set. By extending Sullivan's approach (with some technical complications) Tukia [Tuk97] proved Theorem 33 for Kleinian groups with respect to an arbitrary invariant conformal stream.

Our completely elementary approach subsumes all these particular cases and identifies the conservative part of the boundary action with the big horospheric limit set in full generality, for an arbitrary invariant quasi-conformal stream on a general Gromov hyperbolic space.

It is clear from looking at criterion (*) that the right object in the context of studying conservativity of boundary actions is the big horospheric limit set Λ^{horB} rather than the small one Λ^{horS} . Nonetheless it is plausible that $\Lambda^{\text{horB}} \setminus \Lambda^{\text{horS}}$ is a null set with respect to any invariant quasi-conformal stream on an arbitrary Gromov hyperbolic space. This was proved by Sullivan [Sul81] for Kleinian groups with respect to the visual stream, and for subgroups of a free group (again with respect to the uniform stream) it was done in [GKN07].

We refer the reader to [GKN07] for a recent detailed study of the interrelations between various kinds of limit sets in the simplest model case of the action of a free subgroup on the boundary of the ambient finitely generated free group. Actually, it was this collaboration that brought me to the issues discussed in the present article, and I would like to thank my collaborators Rostislav Grigorchuk and Tatiana Nagnibeda for the gentle insistence with which they encouraged my work.

1. STRUCTURE OF THE ERGODIC COMPONENTS AND RECURRENCE PROPERTIES

1.A. Lebesgue spaces. We begin by recalling the basic properties of the *Lebesgue measure spaces* introduced by Rokhlin, see [Roh52, CFS82]. Measure-theoretically these are the probability spaces such that their non-atomic part is isomorphic to an interval with the Lebesgue measure on it. Thus, any Lebesgue space is uniquely characterized by its *signature* $\sigma = (\sigma_0; \sigma_1, \sigma_2, \dots)$, where σ_0 is the total mass of the non-atomic part, and $\sigma_1 \geq \sigma_2 \geq \dots$ is the ordered sequence of the values of its atoms (extended by zeroes if the set of atoms is finite or empty). There is also an intrinsic definition of Lebesgue spaces based on their separability properties. However, for applications it is usually enough to know that any *Polish topological space* (i.e., separable, metrizable, complete) endowed with a Borel probability measure is a Lebesgue measure space. We shall follow the standard measure theoretical convention:

Unless otherwise specified, all the identities, properties etc. related to measure spaces will be understood mod 0 (i.e., up to null sets). In particular, all the σ -algebras are assumed to be *complete*, i.e., to contain all the measure 0 sets.

An important feature of the Lebesgue measure spaces is

Theorem E (Existence of conditional probabilities). Let $p : (X, m) \rightarrow (\overline{X}, \overline{m})$ be a *homomorphism* (*projection, factorization, quotient map*) of Lebesgue spaces, i.e., for any measurable set $\overline{A} \subset \overline{X}$ its preimage $A = p^{-1}(\overline{A}) \subset X$ is also measurable, and

$m(A) = \overline{m}(\overline{A})$. Then the preimages $X_{\overline{x}} = p^{-1}(\overline{x})$, $\overline{x} \in \overline{X}$, can be uniquely endowed with *conditional* probability measures $m_{\overline{x}}$ in such a way that $(X_{\overline{x}}, m_{\overline{x}})$ are Lebesgue spaces and the measure m decomposes into an integral of the measures $m_{\overline{x}}$, $\overline{x} \in \overline{X}$, with respect to the *quotient measure* \overline{m} on \overline{X} . Namely, for any function $f \in L^1(X, m)$ its restrictions $f_{\overline{x}}$ to $X_{\overline{x}}$ are measurable and belong to the respective spaces $L^1(X_{\overline{x}}, m_{\overline{x}})$, the integrals $\overline{f}(\overline{x}) = \langle f_{\overline{x}}, m_{\overline{x}} \rangle$ depend on \overline{x} measurably, and $\langle f, m \rangle = \langle \overline{f}, \overline{m} \rangle$ (cf. the classical Fubini theorem).

In fact, the above property follows from the classical Fubini theorem in view of Rokhlin's

Theorem C (Classification of homomorphisms of Lebesgue spaces). Any homomorphism $p : (X, m) \rightarrow (\overline{X}, \overline{m})$ of Lebesgue spaces is uniquely (up to an isomorphism) determined by the signatures of the quotient measure \overline{m} and of the conditional measures $m_{\overline{x}}$. Namely, let us denote by I the unit interval endowed with the Lebesgue measure λ , and partition I into a union of consecutive intervals I_0, I_1, \dots with $\lambda(I_i) = \sigma_i$ for a certain signature σ . Further, let us consider a coordinatewise measurable assignment of signatures σ^x to points $x \in I$ which is constant on the intervals I_1, I_2, \dots , and, as before, let I_0^x, I_1^x, \dots be the consecutive subintervals of I with $\lambda(I_i^x) = \sigma_i^x$. Denote by (X, m) the Lebesgue space obtained from the square $(I \times I, \lambda \otimes \lambda)$ by collapsing the sets $\{x\} \times I_i^x$, $i \geq 1, x \in I_0$, and $I_j \times I_i^x$, $i, j \geq 1, x \in I_j$, onto single points. In the same way, let $(\overline{X}, \overline{m})$ be the quotient space of the interval (I, λ) obtained by collapsing the intervals I_1, I_2, \dots onto single points, so that the signature of $(\overline{X}, \overline{m})$ is σ . Then the projection of the square $I \times I$ onto the first coordinate gives rise to a homomorphism from (X, m) to $(\overline{X}, \overline{m})$, and the signatures of the associated conditional measures are precisely σ^x . The claim is that any homomorphism of Lebesgue spaces can be obtained in this way. In particular, if both the quotient measure \overline{m} and all the conditional measures $m_{\overline{x}}$ are purely non-atomic (i.e., have the signature $(1; 0, 0, \dots)$), then the corresponding quotient map is isomorphic just to the projection of the unit square onto the first coordinate.

Obviously, any homomorphism of Lebesgue spaces gives rise to the *preimage sub- σ -algebra* in X which consists of the preimages of all the measurable sets in \overline{X} . Another important feature of the Lebesgue spaces is that, in fact, an arbitrary sub- σ -algebra in X can be obtained in this way for a certain uniquely defined quotient map.

Below we shall use the following elementary fact which follows at once from the uniqueness of the system of conditional measures.

Lemma 1. *Let T be an invertible measure class preserving transformation of a Lebesgue space (X, m) , and let $p : (X, m) \rightarrow (\overline{X}, \overline{m})$ be its T -invariant projection, i.e., $p(Tx) = p(x)$ for a.e. $x \in X$. Then the conditional measures $m_{\overline{x}}$, $\overline{x} \in \overline{X}$, of the projection p are quasi-invariant with respect to T and have the same Radon–Nikodym derivatives as the measure m :*

$$\frac{dTm}{dm}(x) = \frac{dTm_{\overline{x}}}{dm_{\overline{x}}}(x), \quad \text{where } \overline{x} = p(x),$$

for a.e. $x \in X$.

1.B. Ergodic components, continuity and discontinuity. Let now $G \curvearrowright (X, m)$ be an action of an infinite countable group G by measure class preserving transformations on a Lebesgue space (X, m) — which will be our standing assumption through the rest of this Section.

The quotient space $(\overline{X}, \overline{m})$ of (X, m) determined by the σ -algebra of G -invariant sets is called the *space of ergodic components* of the action of G on the space (X, m) , and

the preimages $X_{\bar{x}}$ endowed with the conditional measures $m_{\bar{x}}$ are called the *ergodic components*. The ergodic components $X_{\bar{x}}$ are G -invariant, the conditional measures $m_{\bar{x}}$ are G -quasi-invariant, and the action of G on the spaces $(X_{\bar{x}}, m_{\bar{x}})$ is ergodic (e.g., see [Sch77]). Lemma 1 implies that the conditional measures $m_{\bar{x}}$ have the same Radon–Nikodym derivatives with respect to the action of G as the original measure m .

Since the ergodic components are ergodic, each of them is either purely atomic (in which case it consists of a single G -orbit), or purely non-atomic.

Definition 2. The *continual* \mathcal{C} (resp., *discontinual* \mathcal{D}) *part* of the action $G \curvearrowright (X, m)$ is the union of all the purely non-atomic (resp., purely atomic) components of the action. Denote by $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ the corresponding G -invariant subsets of the space of ergodic components \bar{X} (their measurability follows from Theorem C). An orbit Gx is *continual* (resp., *discontinual*) if it belongs to \mathcal{C} (resp., to \mathcal{D}). The action is *discontinual* if $m(\mathcal{D}) > 0$ and *continual* otherwise. If $m(\mathcal{C}) = 0$ the action is called *completely discontinual*.

Remark 3. The quotient measure class on the space of ergodic components and the measure classes of the conditional measures on the ergodic components do not change when the measure m is replaced with an equivalent one, so that the above Definition (as well as various definitions below related to conservativity and dissipativity) depends only on the measure class of m .

Lemma 4. *Let $A \subset X$ be a measurable G -invariant subset. It is contained in the discontinual part \mathcal{D} if and only if one can select, in a measurable way, a representative from each G -orbit contained in A , i.e., if and only if there exists a measurable map $\pi : A \rightarrow A$ which is constant along the orbits of the action.*

Proof. If π is such a map, then it identifies the space of ergodic components of A with $\pi(A)$, so that in particular $A \subset \mathcal{D}$. Conversely, by Theorem C the discontinual part \mathcal{D} can be identified, in a measurable way, with the product $\bar{\mathcal{D}} \times \{1, 2, \dots\}$. Then one can define the map $\pi : \mathcal{D} \rightarrow \mathcal{D}$ with the required properties as $\pi(\bar{x}, n) = (\bar{x}, 1)$. \square

Remark 5. Below we shall also encounter the situation when instead of a map with the properties from the above Lemma one has an orbit constant measurable map $x \mapsto M_x$, where M_x is a non-empty finite subset of the orbit Gx . This situation can be easily reduced to Lemma 4 by choosing (in a measurable way!) just a single point from each of the sets M_x . This can be done, for instance, by identifying the space (X, m) with the unit interval (with possible collapsing corresponding to the atoms of the measure m) and taking then the minimal of the points of M_x .

1.C. Recurrent and wandering sets. Let us first remind the definitions (e.g., see [Kre85] for the case when G is the group of integers \mathbb{Z}). Actually, we have to slightly modify them (and to distinguish recurrence from infinite recurrence) in order to take into account certain effects which do not arise for the group \mathbb{Z} (see below the remarks after Theorem 11).

Definition 6. A measurable set $A \subset X$ is called *recurrent* (resp., *infinitely recurrent*) if for a.e. point $x \in A$ the trajectory Gx eventually returns to A , i.e., $gx \in A$ for a certain element $g \in G$ other than the group identity e (resp., returns to A infinitely often, i.e., $gx \in A$ for infinitely many elements $g \in G$). The opposite notion is that of a *wandering set* (kind of a “fundamental domain”), i.e., a measurable set $A \subset X$ with pairwise disjoint translates gA , $g \in G$.

We shall now explain the connection between these notions and our Definition 2.

Proposition 7. *Any measurable subset of the continual part \mathcal{C} is infinitely recurrent.*

Proof. For a measurable subset $A \subset \mathcal{C}$ put

$$A_0 = \{x \in A : gx \in A \text{ for finitely many } g \in G\} \subset F,$$

where

$$F = \{x \in X : Gx \cap A \text{ is non-empty and finite}\}.$$

The set F is G -invariant and measurable, and by Lemma 4 and Remark 5 $F \subset \mathcal{D}$. Therefore A_0 must be a null set, whence the claim. \square

Denote by $\mathcal{D}_{\text{free}}$ (resp., $\mathcal{D}_{\text{cofinite}}$) the union of all the *free* (resp., *cofinite*) discontinual orbits, i.e., such that the stabilizers of their points are trivial (resp., finite). Obviously, $\mathcal{D}_{\text{free}} \subset \mathcal{D}_{\text{cofinite}}$, and both these sets are measurable. Let $\overline{\mathcal{D}}_{\text{free}}$ and $\overline{\mathcal{D}}_{\text{cofinite}}$ be the corresponding subsets of the space of ergodic components \overline{X} . The definition immediately implies

Proposition 8. *Any measurable subset of $\mathcal{D} \setminus \mathcal{D}_{\text{free}}$ (resp., of $\mathcal{D} \setminus \mathcal{D}_{\text{cofinite}}$) is recurrent (resp., infinitely recurrent).*

Let us now look at the wandering sets.

Proposition 9. *Any wandering set is contained in $\mathcal{D}_{\text{free}}$, and there is a maximal wandering set, i.e., such that $\mathcal{D}_{\text{free}} = \bigcup_{g \in G} gA$.*

Proof. If A is a wandering set, then the map π from the G -invariant union $\tilde{A} = \bigcup_{g \in G} gA$ to A defined as $\pi(x) = Gx \cap A$ is measurable and G -invariant, so that $A \subset \mathcal{D}$ by Lemma 4. Moreover, all the orbits intersecting A are obviously free, whence $A \subset \mathcal{D}_{\text{free}}$.

As for the maximality, one can take for such a wandering set any measurable section of the projection $\mathcal{D}_{\text{free}} \rightarrow \overline{\mathcal{D}}_{\text{free}}$ (which exists by Lemma 4). \square

1.D. Hopf decomposition.

Definition 10. An action $G \curvearrowright (X, m)$ is called *conservative* (resp., *infinitely conservative*) if any measurable subset $A \subset X$ is recurrent (resp., infinitely recurrent). It is called *dissipative* if there is a non-trivial wandering set, and *completely dissipative* if the whole action space X is the union of translates of a certain wandering set.

The action on the disjoint union of two G -invariant sets is conservative (resp., infinitely conservative) if and only if the action on each of these sets has the same property. Taking stock of the Propositions from Section 1.C we now obtain

Theorem 11 (Hopf decomposition for general actions). *The action space X can be decomposed into the disjoint union of two G -invariant measurable sets (called its conservative and dissipative parts, respectively)*

$$X = [\mathcal{C} \cup (\mathcal{D} \setminus \mathcal{D}_{\text{free}})] \sqcup \mathcal{D}_{\text{free}}$$

such that the restriction of the action to $\mathcal{C} \cup (\mathcal{D} \setminus \mathcal{D}_{\text{free}})$ is conservative and the restriction to $\mathcal{D}_{\text{free}}$ is totally dissipative.

Corollary 12. *If the action $G \curvearrowright (X, m)$ is free, i.e., $\mathcal{D} = \mathcal{D}_{\text{free}}$, then its conservative part coincides with the continual set \mathcal{C} , and its dissipative part coincides with the discontinual set \mathcal{D} .*

Corollary 13 (Poincaré recurrence theorem). *If the measure m is invariant, then $\mathcal{D}_{\text{free}}$ is a null set, and therefore the action is conservative.*

Remark 14. The decomposition into the conservative and totally dissipative parts described in the above Theorem is unique. Indeed, let $X = C_1 \sqcup D_1 = C_2 \sqcup D_2$ be two such decompositions. If they are different, then one of the sets $C_1 \cap D_2, C_2 \cap D_1$ must be non-empty. Let it be, for instance, $A = C_1 \cap D_2$. Then the restriction of the action to A has to be simultaneously conservative (because $A \subset C_1$) and totally dissipative (because $A \subset D_2$), which is impossible.

Remark 15. If the conservativity is replaced with the infinite conservativity, then, generally speaking, the Hopf decomposition as above is not possible. Namely, the action is infinitely conservative on $\mathcal{C} \cup (\mathcal{D} \setminus \mathcal{D}_{\text{cofinite}})$ and is completely dissipative on $\mathcal{D}_{\text{free}}$, whereas on the remaining set $\mathcal{D}_{\text{cofinite}} \setminus \mathcal{D}_{\text{free}}$ it is neither infinitely conservative nor dissipative.

Remark 16. The group \mathbb{Z} does not contain non-trivial finite subgroups, so that for its actions $\mathcal{D}_{\text{cofinite}} = \mathcal{D}_{\text{free}}$, and therefore infinite conservativity is equivalent to plain conservativity. The conservative part of the action in this case is the union of the continual part \mathcal{C} and the set of all the periodic points $\mathcal{D} \setminus \mathcal{D}_{\text{cofinite}}$.

Remark 17. When dealing with the actions of the group \mathbb{Z} one sometimes defines the notion of recurrence by looking only at the “positive semi-orbits” \mathbb{Z}_+x . All measurable subsets of \mathcal{C} are recurrent in this sense as well. Indeed, for a subset $A \subset \mathcal{C}$ let $A_0 = \{x \in A : \mathbb{Z}_+x \cap A \text{ is finite}\}$. Then $x \mapsto zx$, where z is the maximal element of \mathbb{Z} with $zx \in A$, is a measurable “selection map” in the sense of Lemma 4, so that $A_0 \subset \mathcal{D}$, whence A_0 is a null set.

Remark 18. If one defines *strict recurrence* and *strict conservativity* by requiring that the orbit Gx returns to the set A at an orbit point *different* from the starting point x , then the strictly conservative part of the action coincides just with the continual part \mathcal{C} . Again, as with the infinite conservativity, the action on the set $\mathcal{D} \setminus \mathcal{D}_{\text{free}}$ will be neither strictly conservative nor dissipative.

1.E. A continuity criterion.

Theorem 19. *Let $G \curvearrowright (X, m)$ be a free measure class preserving action of a countable group G on a Lebesgue space. Denote by $\mu_x, x \in X$, the measure on the orbit Gx defined as*

$$\mu_x(gx) = \frac{dg^{-1}m}{dm}(x) = \frac{dm(gx)}{dm(x)}$$

(obviously, the measures μ_x corresponding to different points x from the same G -orbit are proportional). Then for a.e. point $x \in X$ the following conditions are equivalent:

- (i) *The orbit Gx is dissipative;*
- (ii) *The orbit Gx is discontinual;*
- (iii) *The measure μ_x is finite;*
- (iv) *For any $t > 0$ the set $\{y \in Gx : \mu_x(y) \geq t\}$ is finite;*
- (v) *The set M_x of maximal weight atoms of the measure μ_x is non-empty and finite.*

Proof. (i) \iff (ii). This is Corollary 12.

(ii) \implies (iii). By Definition 2, the orbit Gx is discontinual if and only if it is an ergodic component of the G -action on X . By Lemma 1 in this case the measure π_x is proportional to the conditional measure on this ergodic component, and therefore it is finite.

(iii) \implies (iv) \implies (v). Obvious.

(v) \implies (i). Follows from Lemma 4 and Remark 5 (because the map $x \mapsto M_x$ is measurable in view of Theorem C). \square

Corollary 20. *Under conditions of Theorem 19 the continual part \mathcal{C} and the discontinual part \mathcal{D} of the action coincide (mod 0) with the sets*

$$(21) \quad \left\{ x \in X : \sum_{g \in G} \frac{dgm(x)}{dm(x)} = \infty \right\}$$

and

$$(22) \quad \left\{ x \in X : \sum_{g \in G} \frac{dgm(x)}{dm(x)} < \infty \right\},$$

respectively.

Remark 23. Theorem 19 (except for the equivalence (i) \iff (ii)) and Corollary 20 (with the summation taken over the equivalence class of x) are also true for an arbitrary *countable non-singular equivalence relation* on a Lebesgue space (X, m) . In this case μ_x are the measures on the equivalence classes determined by the *Radon–Nikodym cocycle* of the equivalence relation. In particular, the equivalence of conditions (ii), (iii), (iv), (v) from Theorem 19 also holds for non-free actions. In what concerns Corollary 20, the only difference with the free case is that one has to replace the summation over g in formulas (21) and (22) with the summation over the orbit of x . The proof is precisely the same; however, in order to spare the reader the trouble of going through the definitions from the ergodic theory of equivalence relations (see [FM77]) we confine ourselves just to free actions. This generality is sufficient, on one hand, to expose our (very simple) line of argument, and, on the other hand, to deal with our main application to boundary actions (Theorem 33).

Remark 24. Condition (iv) from Theorem 19 is *not*, generally speaking, equivalent just to *existence* of $t > 0$ such that the set $\{y \in Gx : \mu_x(y) \geq t\}$ is finite (i.e., to boundedness of the values of the weights of the measure μ_x). The most manifest example of this is an action with a finite invariant measure, see Corollary 13.

2. APPLICATION TO BOUNDARY ACTIONS

2.A. Hyperbolic spaces and limit sets. Recall that a non-compact complete proper metric space \mathcal{X} is *Gromov δ -hyperbolic* (with $\delta \geq 0$) if its metric d satisfies the *δ -ultrametric inequality*

$$(x|z)_o \geq \min\{(x|y)_o, (y|z)_o\} - \delta \quad \forall x, y, z, o \in \mathcal{X},$$

where

$$(x|y)_o = \frac{1}{2} [d(o, x) + d(o, y) - d(x, y)]$$

is the *Gromov product*. In addition we require that the space X be *separable*. On the other hand, we do *not* require the space \mathcal{X} to be *geodesic*. This class of spaces contains Cartan–Hadamard manifolds with pinched sectional curvatures (in particular, the classical hyperbolic spaces of constant negative curvature) and metric trees, see [Gro87, GdlH90] for more details.

A Gromov hyperbolic space \mathcal{X} admits a natural *hyperbolic compactification* $\overline{\mathcal{X}} = \mathcal{X} \cup \partial\mathcal{X}$, and the action of the isometry group $\text{Iso}(\mathcal{X})$ extends by continuity to a continuous *boundary action* on $\partial\mathcal{X}$. The boundary $\partial\mathcal{X}$ is a Polish space.

The *limit set* $\Lambda = \Lambda_G \subset \partial\mathcal{X}$ of a discrete subgroup $G \subset \text{Iso}(\mathcal{X})$ (any such subgroup is at most countable) is the set of all the limit points of any given orbit $Go, o \in \mathcal{X}$, with respect to the hyperbolic compactification, so that the closure of the orbit Go in the

hyperbolic compactification is $Go \cup \Lambda_G$ (this definition does not depend on the choice of the basepoint o). The limit set is closed and G -invariant. Moreover, the action of G on Λ_G is *minimal* (there are no proper G -invariant closed subsets), whereas the action of G on the complement $\partial\mathcal{X} \setminus \Lambda_G$ is *properly discontinuous* (no orbit has accumulation points) [Gro87, Bou95].

The latter result provides a topological decomposition of the boundary action. On the other hand, the situation is more complicated from the measure-theoretical point of view. Let m be a G -quasi-invariant measure on $\partial\mathcal{X}$. Without loss of generality we can assume that it is purely non-atomic. Then, since any element of G fixes at most two boundary points, the action is free. The complement $\partial\mathcal{X} \setminus \Lambda_G$ is obviously contained in the dissipative (\equiv discontinual) part of the action. However, this is as much as can *a priori* be said about the ergodic properties of the boundary action. In particular, the action on Λ_G need not be ergodic or conservative. There are numerous examples witnessing to this; see [GKN07] for a detailed discussion of the simplest model case of the action of a subgroup of a free group on the boundary of the ambient group and for further references.

One can specialize the type of convergence in the definition of the limit set. For instance, the *radial limit set* Λ^{rad} is the set of all the accumulation points of any fixed orbit Go , $o \in \mathcal{X}$, which stay inside a *tubular neighbourhood* of a certain geodesic ray in \mathcal{X} . Yet another type of the boundary convergence, which we are going to describe below, is provided by *horospheric neighborhoods*.

Denote by

$$\beta_z(x, y) = d(y, z) - d(x, z) \quad x, y \in \mathcal{X}$$

the *distance cocycle* associated with a point $z \in \mathcal{X}$, and, following [Kai04], put

$$(25) \quad \beta_\omega(x, y) = \limsup_{z \rightarrow \omega} \beta_z(x, y) \quad \forall x, y \in \mathcal{X}, \omega \in \partial\mathcal{X}.$$

If the space \mathcal{X} is CAT(-1) (e.g., a Cartan–Hadamard manifold with pinched sectional curvatures or a tree), then \limsup in the above formula can be replaced just with the ordinary limit, and β_ω are the boundary *Busemann cocycles*. Although for a general Gromov hyperbolic space β_ω are not, generally speaking, cocycles, they still satisfy the cocycle identity with a uniformly bounded error (i.e., they are *quasi-cocycles*). Namely,

Proposition 26 ([Kai04]). *There exists a constant $C \geq 0$ depending on the hyperbolicity constant δ of the space \mathcal{X} only such that for any $\omega \in \partial\mathcal{X}$ the function β_ω (25) has the following properties:*

- (i) β_ω is “jointly Lipschitz”, i.e., $|\beta_\omega(x, y)| \leq d(x, y)$ for all $x, y \in \mathcal{X}$, in particular, $\beta_\omega(x, x) \equiv 0$;
- (ii) $0 \leq \beta_\omega(x, y) + \beta_\omega(y, z) + \beta_\omega(z, x) \leq C$ for all $x, y, z \in \mathcal{X}$.

The quasi-cocycles β_ω are obviously invariant with respect to the isometries of \mathcal{X} , i.e.,

$$\beta_{g\omega}(gx, gy) = \beta_\omega(x, y) \quad \forall x, y \in \mathcal{X}, \omega \in \partial\mathcal{X}, g \in \text{Iso}(\mathcal{X}).$$

We shall define the *horoball* in \mathcal{X} centered at a boundary point $\omega \in \partial\mathcal{X}$ and passing through a point $o \in \mathcal{X}$ as

$$\text{HBall}_\omega(o) = \{x \in \mathcal{X} : \beta_\omega(o, x) \leq 0\}.$$

Definition 27. The *big* (resp., *small*) *horospheric limit set* $\Lambda^{\text{horB}} = \Lambda_G^{\text{horB}}$ (resp., $\Lambda^{\text{horS}} = \Lambda_G^{\text{horS}}$) of a discrete group G of isometries of a Gromov hyperbolic space \mathcal{X} is the set of all the points $\omega \in \partial\mathcal{X}$ such that a certain (resp., any) horoball centered at ω contains infinitely many points from a fixed orbit Go , $o \in \mathcal{X}$ (the resulting set does not depend on the choice of the orbit Go , see Remark 28 below).

Remark 28. As it follows from Proposition 26, for any fixed reference point $o \in \mathcal{X}$ the big (resp., small) horospheric limit sets can also be defined as the set of all the points $\omega \in \partial\mathcal{X}$ for which the set

$$\{x \in Go : \beta_\omega(o, x) \leq t\}$$

is infinite for a certain (resp., for any) $t \in \mathbb{R}$.

Remark 29. Usually our small horospheric limit set is called just the *horospheric limit set*, and in the context of Fuchsian groups its definition, along with the definition of the radial limit set, goes back to Hedlund [Hed36]. Following [Mat02] (in the Kleinian case) we call it *small* in order to better distinguish it from the *big* one, which, although apparently first explicitly introduced by Tukia [Tuk97] (again just in the Kleinian case), essentially appears (for Fuchsian groups) already in Pommerenke's paper [Pom76].

The horospheric limit sets $\Lambda^{\text{horS}}, \Lambda^{\text{horB}}$ are obviously G -invariant, Borel, and contained in the full limit set Λ (because the only boundary accumulation point of any horoball is just its center).

2.B. Boundary conformal streams.

Definition 30. A family of pairwise equivalent finite measures $\lambda = \{\lambda_x\}$ on the boundary $\partial\mathcal{X}$ of a Gromov hyperbolic space \mathcal{X} parameterized by points $x \in \mathcal{X}$ is called a *quasi-conformal stream* of dimension $D > 0$ if there exists a constant $C > 0$ such that

$$\left| \log \frac{d\lambda_x}{d\lambda_y}(\omega) - D\beta_\omega(x, y) \right| \leq C \quad \forall x, y \in \mathcal{X}, \omega \in \partial\mathcal{X}.$$

A stream λ is *invariant* with respect to a group $G \subset \text{Iso}(\mathcal{X})$ if

$$\lambda_{gx} = g\lambda_x \quad \forall g \in G, x \in \mathcal{X}.$$

Remark 31. We follow here the terminology developed in [KL05]. More traditionally, any invariant quasi-conformal stream is determined just by a single finite boundary *quasi-conformal measure* $\lambda = \lambda_o$ with the property that

$$(32) \quad \left| \frac{dg\lambda}{d\lambda}(\omega) - D\beta_\omega(g o, o) \right| \leq C \quad \forall g \in G, \omega \in \partial\mathcal{X}.$$

for a certain reference point $o \in \mathcal{X}$. If β_ω are cocycles (which is the case for $\text{CAT}(-1)$ spaces), then any measure λ satisfying (32) is equivalent to a unique finite measure λ' (called *conformal*) which satisfies formula (32) with $C = 0$ (it follows from the fact that any uniformly bounded cocycle is cohomologically trivial). This definition is motivated by the fact that the *visual measure* on the boundary sphere $\partial\mathbb{H}^{d+1}$ of the classical $(d+1)$ -dimensional hyperbolic space with sectional curvature -1 is conformal of dimension d (in our terminology it means that the *visual stream* which consists in assigning to any point from the hyperbolic space the associated visual measure is conformal). However, the limit set of the group can be “much smaller” than the boundary sphere and be a null set with respect to the visual measure. Existence of conformal measures which are concentrated on the limit set (and for which the dimension coincides with the critical exponent of the group) was first established by Patterson [Pat76] in the case of Fuchsian groups. His construction was further generalized (see [Sul79], [Kai90]), ultimately providing existence of a conformal measure for any closed subgroup of isometries of a general $\text{CAT}(-1)$ space [BM96]. For discrete isometry groups of general Gromov hyperbolic spaces existence of measures satisfying (32) (i.e., existence of invariant quasi-conformal streams in our terminology) was established by Coornaert [Coo93] (also by a generalization of Patterson's construction).

We are now ready to proceed to the main application of Theorem 19 which is a description of the Hopf decomposition of (the measure class of) a quasi-conformal stream invariant with respect to a discrete subgroup $G \subset \text{Iso}(\mathcal{X})$. As the atomic part of such a stream is obviously discontinual (so that by Theorem 33 its Hopf decomposition is completely determined by the size of the point stabilizers), we can restrict our considerations to the purely non-atomic case only.

Theorem 33. *Let G be a discrete group of isometries of a Gromov hyperbolic space \mathcal{X} . Then the big horospheric limit set Λ_G^{horB} is (mod 0) the conservative part of the boundary action of G with respect to any purely non-atomic G -invariant boundary quasi-conformal stream λ .*

Proof. Fix a reference point $o \in \mathcal{X}$ and consider the associated measure λ_o . Without loss of generality we may assume that it is normalized, so that $(\partial\mathcal{X}, \lambda_o)$ is a Lebesgue space. Any isometry of \mathcal{X} has at most two fixed boundary points, the group G is countable, and the measure m is purely non-atomic. Therefore, the action of G on the space $(\partial\mathcal{X}, \lambda_o)$ is free, and we can apply Theorem 19. By condition (iv) the orbit $G\omega$ is dissipative if and only if all the sets $\{g \in G : dg\lambda_o/d\lambda_o(\omega) \geq t\}$ are finite, which, in view of (32), is the same as the finiteness of all the sets $\{g \in G : \beta_\omega(g\omega, o) \geq t\}$, i.e., the same as the finiteness of the intersection of the orbit $G\omega$ with any horoball centered at ω . Thus, the orbit $G\omega$ is dissipative if and only if it is contained in the complement of Λ_G^{horB} . \square

Corollary 34. *By Corollary 20, the big horospheric limit set Λ_G^{horB} (\equiv the conservative part of the boundary action) coincides with the divergence set of the Poincaré–Busemann series*

$$\left\{ \omega \in \partial\mathcal{X} : \sum_{g \in G} e^{D\beta_\omega(g\omega, o)} = \infty \right\}.$$

Remark 35. For Fuchsian groups with respect to the visual stream on the boundary circle Theorem 33 and Corollary 34 were proved by Pommerenke [Pom76] (although in a somewhat different terminology, see the discussion in [Pom82, Section 1]). Pommerenke’s argument uses analytic properties of the Blaschke products and does not immediately carry over to the higher dimensional situation. Sullivan [Sul81] used a more direct geometrical approach and established Theorem 33 for Kleinian groups, again with respect to the visual stream (actually he considered the small horospheric limit set Λ^{horS} essentially showing that $\Lambda^{\text{horB}} \setminus \Lambda^{\text{horS}}$ is a null set, see the Remark below). By extending Sullivan’s approach (with some technical complications) Tukia [Tuk97] proved Theorem 33 for Kleinian groups with respect to an arbitrary invariant conformal stream.

Remark 36. Our argument used the characterization of the conservative part of a free action $G \curvearrowright (X, m)$ as the set of all the points $x \in X$ such that for a *certain* $t > 0$

$$\{g \in G : dgm/dm(x) > t\} \text{ is infinite}$$

(see condition (iv) from Theorem 19), which in our setup is precisely the big horospheric limit set Λ^{horB} . The small horospheric limit set Λ^{horS} corresponds to requiring that the above condition hold for *any* $t > 0$, which, in general, is *not* equivalent to conservativity (see Remark 24). From this point of view the right object in the context of studying conservativity of boundary actions is definitely Λ^{horB} rather than Λ^{horS} . The difference $\Lambda^{\text{horB}} \setminus \Lambda^{\text{horS}}$ is the set of all the boundary points $\omega \in \partial\mathcal{X}$ for which among the horoballs centered at ω there are both ones containing *finitely many points* from the orbit $G\omega$ of a fixed reference point $o \in \mathcal{X}$ and ones containing *infinitely many points* from $G\omega$. Sullivan

[Sul81] essentially proved that $\Lambda^{\text{horB}} \setminus \Lambda^{\text{horS}}$ is a null set with respect to the visual stream for Kleinian groups (also see the discussion of his result in [Pom82, Section 1]); for subgroups of a free group (again with respect to the uniform stream on the boundary of the ambient group) it was done in [GKN07]. We are not aware of any other results of this kind; in particular, it is already not known for Kleinian groups with respect to general invariant conformal streams, see [Tuk97]. Nonetheless, it seems plausible that $\Lambda^{\text{horB}} \setminus \Lambda^{\text{horS}}$ is a null set with respect to any invariant quasi-conformal stream on an arbitrary Gromov hyperbolic space.

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